M1

Time: 3 Hours

Number of Questions: 24

Max Marks: 100

INSTRUCTIONS

1. Use of mobile phones, smartphones, iPads, calculators, programmable wrist watches is **STRICTLY PROHIBITED.** Only ordinary pens and pencils are allowed inside the examination hall.

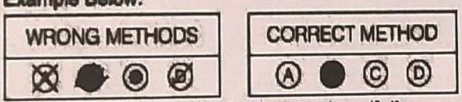
2. The correction is done by machines through scanning. On the OMR Sheet, darken bubbles completely with a **black or blue ball pen**. Please **DO NOT use a pencil or a gel pen.** Darken the bubbles completely, only after you are sure of your answer; else, erasing may lead to the OMR sheet getting damaged and the machine may not be able to read the answer.

3. Incompletely, incorrectly or carelessly filled information may disqualify your candidature.

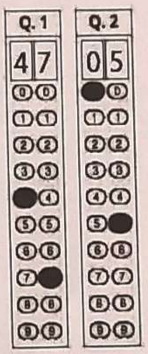
4. Each question has a one or two digit number as answer. The first diagram below shows improper and proper way of darkening the bubbles with detailed instructions. The second diagram shows how to mark a 2-digit number and a 1-digit number.

INSTRUCTIONS

1. "Think before you ink".
2. Marking should be done with Blue/Black Ball Point Pen only.
3. Darken only one circle for each question as shown in Example Below.



4. If more than one circle is darkened or if the response is marked in any other way as shown "WRONG" above, it shall be treated as wrong way of marking.
5. Make the marks only in the spaces provided.
6. Carefully tear off the duplicate copy of the OMR without tampering the Original.
7. Please do not make any stray marks on the answer sheet.



5. The answer you write on OMR sheet is irrelevant. The darkened bubble will be considered as your final answer.

6. Questions 1 to 10 carry 2 marks each; questions 11 to 22 carry 5 marks each; questions 23 and 24 carry 10 marks each.

7. All questions are compulsory.

8. There are no negative marks.

9. Do all rough work in the space provided below for it. You also have blank pages at the end of the question paper to continue with rough work.

10. After the exam, you may take away the Candidate's copy of the OMR sheet.

11. Preserve your copy of OMR sheet till the end of current olympiad season. You will need it later for verification purposes.

12. You may take away the question paper after the examination.

1. A triangle ABC with $AC = 20$ is inscribed in a circle ω. A tangent t to ω is drawn through B. The distance of t from A is 25 and that from C is 16. If S denotes the area of the triangle ABC, find the largest integer not exceeding $S/20$.

2. In a parallelogram $ABCD$, a point P on the segment AB is taken such that $\dfrac{AP}{AB} = \dfrac{61}{2022}$ and a point Q on the segment AD is taken such that $\dfrac{AQ}{AD} = \dfrac{61}{2065}$. If PQ intersects AC at T, find $\dfrac{AC}{AT}$ to the nearest integer.

3. In a trapezoid $ABCD$, the internal bisector of angle A intersects the base BC (or its extension) at the point E. Inscribed in the triangle ABE is a circle touching the side AB at M and side BE at the point P. Find the angle DAE in degrees, if $AB : MP = 2$.

4. Starting with a positive integer M written on the board, Alice plays the following game: in each move, if x is the number on the board, she replaces it with $3x + 2$. Similarly, starting with a positive integer N written on the board, Bob plays the following game: in each move, if x is the number on the board, he replaces it with $2x + 27$. Given that Alice and Bob reach the same number after playing 4 moves each, find the smallest value of $M + N$.

5. Let m be the smallest positive integer such that $m^2 + (m+1)^2 + \cdots + (m+10)^2$ is the square of a positive integer n. Find $m + n$.

6. Let a, b be positive integers satisfying $a^3 - b^3 - ab = 25$. Find the largest possible value of $a^2 + b^3$.

7. Find the number of ordered pairs (a, b) such that $a, b \in \{10, 11, \cdots, 29, 30\}$ and
$$\mathrm{GCD}(a, b) + \mathrm{LCM}(a, b) = a + b.$$

8. Suppose the prime numbers p and q satisfy $q^2 + 3p = 197p^2 + q$. Write $\dfrac{q}{p}$ as $l + \dfrac{m}{n}$, where l, m, n are positive integers, $m < n$ and $\mathrm{GCD}(m, n) = 1$. Find the maximum value of $l + m + n$.

9. Two sides of an integer sided triangle have lengths 18 and x. If there are exactly 35 possible integer values y such that $18, x, y$ are the sides of a non-degenerate triangle, find the number of possible integer values x can have.

10. Consider the 10-digit number $M = 9876543210$. We obtain a new 10-digit number from M according to the following rule: we can choose one or more disjoint pairs of adjacent digits in M and interchange the digits in these chosen pairs, keeping the remaining digits in their own places. For example, from $M = 9\underline{87}6\underline{54}3210$, by interchanging the 2 underlined pairs, and keeping the others in their places, we get $M_1 = 9\overline{78}6\overline{45}3210$. Note that any number of (disjoint) pairs can be interchanged. Find the number of new numbers that can be so obtained from M.

11. Let AB be a diameter of a circle ω and let C be a point on ω, different from A and B. The perpendicular from C intersects AB at D and ω at $E(\neq C)$. The circle with centre at C and radius CD intersects ω at P and Q. If the perimeter of the triangle PEQ is 24, find the length of the side PQ.

12. Given ΔABC with $\angle B = 60^\circ$ and $\angle C = 30^\circ$, let P, Q, R be points on sides BA, AC, CB respectively such that $BPQR$ is an isosceles trapezium with $PQ \parallel BR$ and $BP = QR$. Find the maximum possible value of $\dfrac{2[ABC]}{[BPQR]}$ where $[S]$ denotes the area of any polygon S.

13. Let ABC be a triangle and let D be a point on the segment BC such that $AD = BC$. Suppose $\angle CAD = x^\circ$, $\angle ABC = y^\circ$ and $\angle ACB = z^\circ$ and x, y, z are in an arithmetic progression in that order where the first term and the common difference are positive integers. Find the largest possible value of $\angle ABC$ in degrees.

14. Let x, y, z be complex numbers such that

$$\frac{x}{y+z} + \frac{y}{z+x} + \frac{z}{x+y} = 9$$
$$\frac{x^2}{y+z} + \frac{y^2}{z+x} + \frac{z^2}{x+y} = 64$$
$$\frac{x^3}{y+z} + \frac{y^3}{z+x} + \frac{z^3}{x+y} = 488$$

If $\dfrac{x}{yz} + \dfrac{y}{zx} + \dfrac{z}{xy} = \dfrac{m}{n}$ where m, n are positive integers with $\mathrm{GCD}(m, n) = 1$, find $m + n$.

15. Let x, y be real numbers such that $xy = 1$. Let T and t be the largest and the smallest values of the expression

$$\frac{(x+y)^2 - (x-y) - 2}{(x+y)^2 + (x-y) - 2}.$$

If $T + t$ can be expressed in the form $\dfrac{m}{n}$ where m, n are nonzero integers with $\mathrm{GCD}(m, n) = 1$, find the value of $m + n$.

16. Let a, b, c be reals satisfying

$$3ab + 2 = 6b, \qquad 3bc + 2 = 5c, \qquad 3ca + 2 = 4a.$$

Let $\mathbb{Q}$ denote the set of all rational numbers. Given that the product abc can take two values $\dfrac{r}{s} \in \mathbb{Q}$ and $\dfrac{t}{u} \in \mathbb{Q}$, in lowest form, find $r + s + t + u$.

17. For a positive integer $n > 1$, let $g(n)$ denote the largest positive proper divisor of n and $f(n) = n - g(n)$. For example, $g(10) = 5, f(10) = 5$ and $g(13) = 1, f(13) = 12$. Let N be the smallest positive integer such that $f(f(f(N))) = 97$. Find the largest integer not exceeding $\sqrt{N}$.

18. Let m, n be natural numbers such that

$$m + 3n - 5 = 2\,\mathrm{LCM}(m, n) - 11\,\mathrm{GCD}(m, n).$$

Find the maximum possible value of $m + n$.

19. Consider a string of n 1's. We wish to place some $+$ signs in between so that the sum is 1000. For instance, if $n = 190$, one may put $+$ signs so as to get 11 ninety times and 1 ten times, and get the sum 1000. If a is the number of positive integers n for which it is possible to place $+$ signs so as to get the sum 1000, then find the sum of the digits of a.

20. For an integer $n \geq 3$ and a permutation $\sigma = (p_1, p_2, \ldots, p_n)$ of $\{1, 2, \ldots, n\}$, we say p_l is a *landmark point* if $2 \leq l \leq n-1$ and $(p_{l-1} - p_l)(p_{l+1} - p_l) > 0$. For example, for $n = 7$, the permutation $(2, 7, 6, 4, 5, 1, 3)$ has four landmark points: $p_2 = 7, p_4 = 4, p_5 = 5$ and $p_6 = 1$. For a given $n \geq 3$, let $L(n)$ denote the number of permutations of $\{1, 2, \ldots, n\}$ with exactly one landmark point. Find the maximum $n \geq 3$ for which $L(n)$ is a perfect square.

21. An ant is at a vertex of a cube. Every 10 minutes it moves to an adjacent vertex along an edge. If N is the number of one hour journeys that end at the starting vertex, find the sum of the squares of the digits of N.

22. A binary sequence is a sequence in which each term is equal to 0 or 1. A binary sequence is called *friendly* if each term is adjacent to at least one term that is equal to 1. For example, the sequence $0, 1, 1, 0, 0, 1, 1, 1$ is friendly. Let F_n denote the number of friendly binary sequences with n terms. Find the smallest positive integer $n \geq 2$ such that $F_n > 100$.

23. In a triangle ABC, the median AD divides $\angle BAC$ in the ratio $1:2$. Extend AD to E such that EB is perpendicular AB. Given that $BE = 3$, $BA = 4$, find the integer nearest to BC^2.

24. Let N be the number of ways of distributing 52 identical balls into 4 distinguishable boxes such that no box is empty and the difference between the number of balls in any two of the boxes is not a multiple of 6. If $N = 100a + b$, where a, b are positive integers less than 100, find $a + b$.